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ON THE THEOREMS OF GAUSS AND GREEN.

BY VINCENT C. POOR.

PART I.

The Divergence of a Vector.

Definitions.—Among the definitions for the divergence of a vector, we have the following:

$$\operatorname{div} \mathbf{u} = \operatorname*{Limit}_{\tau \to 0} \frac{\int \mathbf{u} \times \mathbf{n}\, d\sigma}{\tau},$$

where $\mathbf{u}$ is a continuous vector point function whose scalar product $\mathbf{u} \times \mathbf{n}$ with the outward unit normal $\mathbf{n}$ of the surface σ is integrable over the boundary of the region τ, or over the bounding surface of any sub-region of the region τ. This definition may be looked upon as a differential form of Gauss's Theorem:

$$(1) \qquad \int \operatorname{div} \mathbf{u}\, d\tau = \int \mathbf{u} \times \mathbf{n}\, d\sigma.$$

In the elements of vector analysis, Burali-Forti and Marcolongo give the following definition:

$$\operatorname{div} \mathbf{u} = \{\operatorname{grad} (\mathbf{u} \times \mathbf{a}) - \operatorname{rot} (\mathbf{u} \wedge \mathbf{a})\} \times \mathbf{a},^*$$

where $\mathbf{a}$ is an arbitrary constant vector. The symbol, $\wedge$, read vec, is Burali-Forti and Marcolongo's symbol for the vector product of two vectors, and the abbreviations, grad and rot, are the gradient, and the rotation or curl of a vector respectively.

The former definition is not convenient for proving directly such theorems as

$$\begin{aligned} \operatorname{div} m\mathbf{u} &= m \operatorname{div} \mathbf{u} + \operatorname{grad} m \times \mathbf{u}, \\ \operatorname{div} \mathbf{u} \wedge \mathbf{v} &= \mathbf{u} \times \operatorname{rot} \mathbf{v} - \mathbf{v} \times \operatorname{rot} \mathbf{u} \end{aligned}$$

and the uniqueness and existence of the limit might be difficult to establish without resorting to a coördinate system. The second definition seems very artificial, to say the least, and not of a form to be readily remembered.

Some of these objections are obviated through the following definition, which, expressed in the notation of Burali-Forti and Marcolongo, reads as follows:

$$(2) \quad \operatorname{div} \mathbf{u} \cdot dP \times \delta P \wedge \vartheta P = d\mathbf{u} \times \delta P \wedge \vartheta P + \delta \mathbf{u} \times \vartheta P \wedge dP + \vartheta \mathbf{u} \times dP \wedge \delta P,$$

where the point differentials dP, δP, ϑP are any three non-complanar vectors. This form is both natural and convenient; natural, since it may be easily seen to be a differential form of Gauss' Theorem, and convenient because of the differential operators involved. The expanded forms of div $m\mathbf{u}$ and div $\mathbf{u} \wedge \mathbf{v}$ are readily deducible. By certain changes this form of definition may be deduced from Burali-Forti's definition* for the first invariant of a particular homography.

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* "Elements de Calcul Vectoriel, Burali-Forti et Marcolongo," p. 71, article 3.

Gauss' Theorem.—Also from this definition the integral form of Gauss' Theorem may be established. To show this, the region τ, bounded by the surface σ, may be divided into cells by passing planes parallel to the planes determined by the arbitrary vectors dP, δP, ϑP, taken in pairs. The right member of (2) is the resultant flux of the vector $\mathbf{u}$, through the surface of each cell. The resultant flux of $\mathbf{u}$ through each face common to two adjacent cells is zero, since the outward normals are of opposite sense. Upon applying the fundamental law of the integral calculus the right member in the limit is seen to be the flux integral of the vector $\mathbf{u}$ through the surface σ, and Gauss' Theorem follows.

Uniqueness.—To prove the uniqueness of the idea defined by (2) we can, in the usual way, assume another operator operating on $\mathbf{u}$, div′ $\mathbf{u}$. Then by definition

$$\text{(3)} \quad \operatorname{div}'\mathbf{u}\cdot dP \times \delta P \wedge \vartheta P = d\mathbf{u} \times \delta P \wedge \vartheta P + \delta\mathbf{u} \times \vartheta P \wedge dP + \vartheta\mathbf{u} \times dP \wedge \delta P.$$

Subtracting (3) from (2) we have

$$(\operatorname{div}\mathbf{u} - \operatorname{div}'\mathbf{u})dP \times \delta P \wedge \vartheta P \equiv 0.$$

But since the volume element $dP \times \delta P \wedge \vartheta P$ is arbitrary,

$$\operatorname{div}\mathbf{u} = \operatorname{div}'\mathbf{u}.$$

Existence.—The existence of div $\mathbf{u}$ depends on the existence of $d\mathbf{u}$, which in turn exists if

$$\operatorname*{Limit}_{h \to 0} \frac{\mathbf{u}(P + hdP) - \mathbf{u}(P)}{h}$$

exists.

Or, if $\mathbf{u}$ is a function of the distance r, we may write

$$\begin{aligned}
\operatorname{div}\mathbf{u}\, d\tau &= \frac{\partial\mathbf{u}}{\partial r} dr \times \delta P \wedge \vartheta P + \frac{\partial\mathbf{u}}{\partial r} \delta r \times \vartheta P \wedge dP + \cdots \\
&= \frac{\partial\mathbf{u}}{\partial r} (\operatorname{grad} r \times dP) \times \delta P \wedge \vartheta P + \cdots + \cdots \\
&= \operatorname{grad} r \times \left[\frac{\partial\mathbf{u}}{\partial r} \times \delta P \wedge \vartheta P \cdot dP + \frac{\partial\mathbf{u}}{\partial r} \times \vartheta P \wedge dP \cdot \delta P \right. \\
&\qquad \left. + \frac{\partial\mathbf{u}}{\partial r} \times dP \wedge \delta P \cdot \vartheta P \right] \\
&= \operatorname{grad} r \times \frac{\partial\mathbf{u}}{\partial r} \cdot dP \times \delta P \wedge \vartheta P,
\end{aligned}$$

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* "Transformations Linéaires, Burali-Forti et Marcolongo," p. 23, Article 3.

since the vector $dP \times \delta P \wedge \vartheta P \cdot (\partial \mathbf{u}/\partial r)$ is expressible linearly in terms of the vectors dP, δP, and ϑP, in the form given in square brackets. Hence,

$$\operatorname{div} \mathbf{u} = \operatorname{grad} r \times \frac{\partial \mathbf{u}}{\partial r}\cdot$$

Thus div **u** is seen to exist if $\partial \mathbf{u}/\partial r$ exists. Or, in rectangular coördinates, we may say that the divergence of the vector **u** exists if the partial derivatives of **u**, with respect to x, y, and z, exist.

Example.—Let us apply the definition (1) to the vector point function $\mathbf{r} = P - O$, O being a fixed point. We thus have, since $dr = dP$,

$$\operatorname{div} \mathbf{r} \cdot d\tau = dP \times \delta P \wedge \vartheta P + \delta P \times \vartheta P \wedge dP + \vartheta P \times dP \wedge \delta P.$$

Since each of the scalar triple products in the right member is the same and equal to $d\tau$, this right member becomes $3d\tau$, so that $\operatorname{div} \mathbf{r} = 3$.

PART II.

Green's Theorem.

A General Form.—Turning from the elements to the more advanced part of vector analysis, we find in the "Transformations Linéaires" of Burali-Forti and Marcolongo the divergence of a vector defined as the first invariant of the homography* $d\mathbf{u}/dP$, written

$$\operatorname{div} \mathbf{u} \equiv I_1 \left(\frac{d\mathbf{u}}{dP}\right)\cdot$$

From this definition for div **u**, the following theorem may be obtained:

$$\operatorname{div} \alpha u = I_1 \left(\alpha \frac{d\mathbf{u}}{dP}\right) + \operatorname{grad} K\alpha \times \mathbf{u}, \tag{4}$$

where $K\alpha$ is the conjugate of the homography α. We are now able to demonstrate the following very general form of Green's Theorem:

$$\int I_1 \left(K\alpha \cdot \frac{d \operatorname{grad} \beta}{dP} - K\beta \frac{d \operatorname{grad} \alpha}{dP}\right) d\tau = \int (K\alpha \operatorname{grad} \beta - K\beta \operatorname{grad} \alpha) \times \mathbf{n}\, d\sigma, \tag{5}$$

where α and β are homographies.

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* A homography is defined as any linear operator which transforms a vector into a vector.

In rectangular coördinates, the homography

$$\alpha \equiv \left\| \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} \right\|$$

operating on the vector

$$\mathbf{u} = u_1 i + u_2 j + u_3 k$$

changes the vector u into the vector

$$\alpha\mathbf{u} = (a_{11}u_1 + a_{12}u_2 + a_{13}u_3)\mathbf{i} + (a_{21}u_1 + a_{22}u_2 + a_{23}u_3)\mathbf{j} + (a_{31}u_1 + a_{32}u_2 + a_{33}u_3)\mathbf{k}.$$

Proof.—If we substitute (4) in Gauss' Theorem (1), we will have

$$(6)\qquad \int I_1\left(\alpha\frac{d\mathbf{u}}{dP}\right)d\tau - \int \alpha\mathbf{u}\times\mathbf{n}d\sigma = -\int \text{grad}\ K\alpha\times\mathbf{u}\ d\tau.$$

If we now take $\mathbf{u} = \text{grad}\ K\beta$, (6) becomes

$$(7)\qquad \int I_1\left(\alpha\frac{d\ \text{grad}\ K\beta}{dP}\right)d\tau - \int \alpha\ \text{grad}\ K\beta\times\mathbf{n}\ d\sigma$$
$$= -\int \text{grad}\ K\alpha\times\text{grad}\ K\beta\ d\tau,$$

a form of Green's Theorem. Since the right member of (7) is symmetric in α and β, the left member is likewise, so that α and β may be interchanged without affecting the truth of equation (7). Thus:

$$(8)\qquad \int I_1\left(\beta\frac{d\ \text{grad}\ \alpha}{dP}\right)d\tau - \int \beta\ \text{grad}\ K\alpha\times\mathbf{n}\ d\sigma$$
$$= -\int \text{grad}\ K\alpha\times\text{grad}\ K\beta\ d\tau.$$

Subtracting (8) from (7) and transposing, we have

$$(9)\qquad \int I_1\left(\alpha\frac{d\ \text{grad}\ K\beta}{dP} - \beta\frac{d\ \text{grad}\ K\alpha}{dP}\right)d\tau$$
$$= \int (\alpha\ \text{grad}\ K\beta - \beta\ \text{grad}\ K\alpha)\times\mathbf{n}\ d\sigma.$$

Equation (5) is true for any two homographies. It is therefore true if we replace α and β by $K\alpha$ and $K\beta$. If we make this substitution in (9) and remember that $KK\alpha = \alpha$ we obtain equation (5).

Special Forms.—The operators I_1 and K are linear operators, changing a homography into a homography. Thus

$$I_1(m\alpha) = mI_1(\alpha),$$

if m is a scalar point function. Also $K(m) = m$. If, then, we replace the homographies α and β by the scalar point functions φ and ψ, equation (5) reduces to

$$\int\left[\varphi I_1\left(\frac{d\ \text{grad}\ \psi}{dP}\right) - \psi I_1\left(\frac{d\ \text{grad}\ \varphi}{dP}\right)\right]d\tau = \int \varphi\ \text{grad}\ \psi - \psi\ \text{grad}\ \varphi)\times\mathbf{n}\ d\sigma.$$

If we denote div grad by Δ this last equation may be written

$$\int (\varphi\Delta\psi - \psi\Delta\varphi)d\tau = \int (\varphi\ \text{grad}\ \psi - \psi\ \text{grad}\ \varphi)\times\mathbf{n}\ d\sigma,$$

the well-known form of Green's Theorem, which appears as a special instance of the more general equation (5). Further if α is taken as unity and grad β as the vector $\mathbf{u}$, (5) reduces to Gauss' Theorem as it should.

Another theorem, closely allied to these, but which may better be classed with the theorems in a previous paper* by the writer, will be given without proof. The proof may follow the lines suggested by that paper.

THEOREM.—*If* α *is a homography symmetrical in* P *and* M *such that* $d\alpha/dM = -(d\alpha/dP)$ *and if* $\mathbf{u}$ *is independent of* P*, the point of integration, then:*

$$\int\left(\frac{d\alpha}{dP}\mathbf{u}\right)\mathbf{x}\,d\tau = -\int\mathbf{u}\times\mathbf{n}\,\alpha\mathbf{x}\,d\sigma - \int\alpha\frac{d\mathbf{x}}{dP}\mathbf{u}\,d\tau.$$

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* *Bulletin American Mathematical Society,* Vol. 22, Jan., 1916, p. 174. "Transformations in the Theory of the Linear Vector Function."